EXPERIENCES WITH MULTI-LEVEL SPECTRAL DEFERRED CORRECTION METHODS

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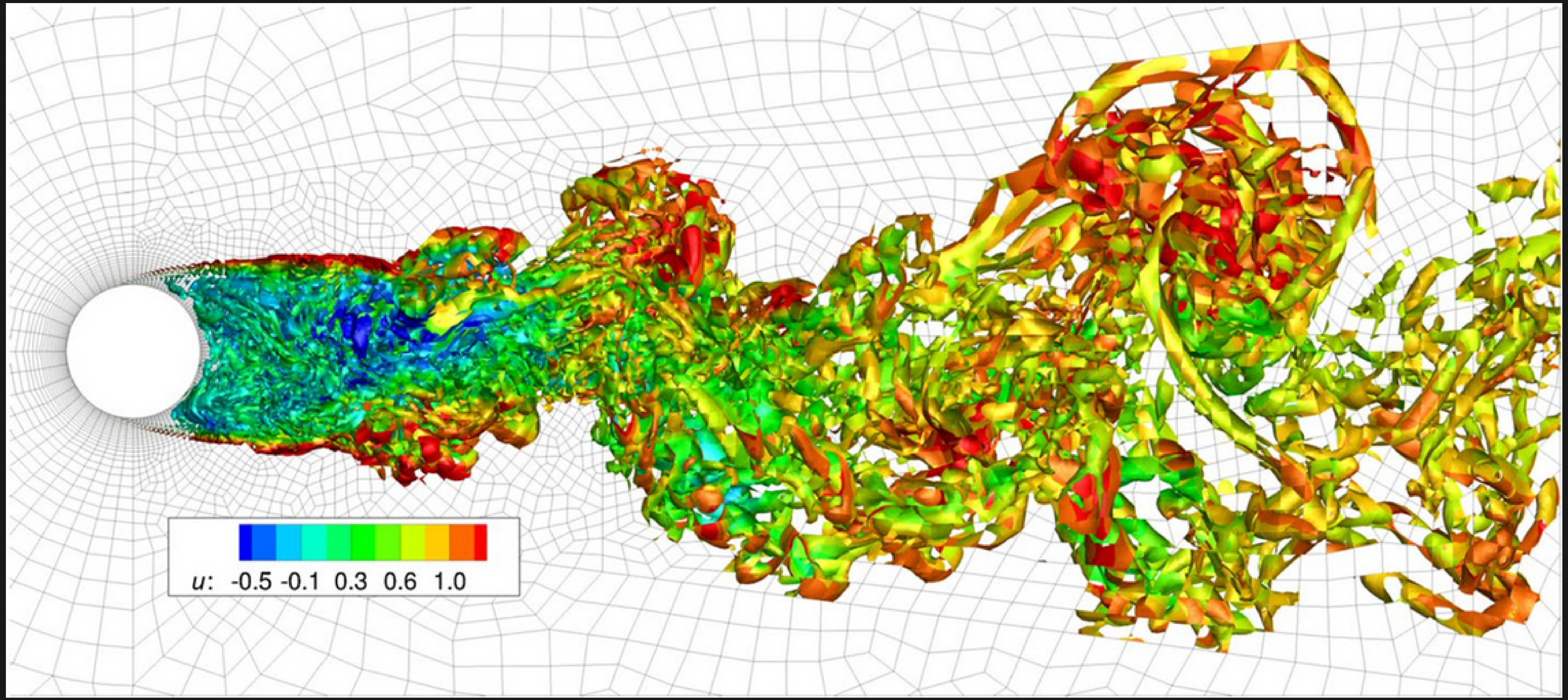
MOTIVATION

- Spatial HOM combined with low order time integrators
- Discrepancy spatial and temporal accuracy

⬇ Solution

- Local adaptivity in space and time
  → more efficiency and runtime savings



Numerical simulation of the flow over a cylinder at Reynolds number 3900: unstructured mesh and instantaneous vortices colored by streamwise velocity [Pan et al. (2021) Pan et. al., 2021]

OUTLINE

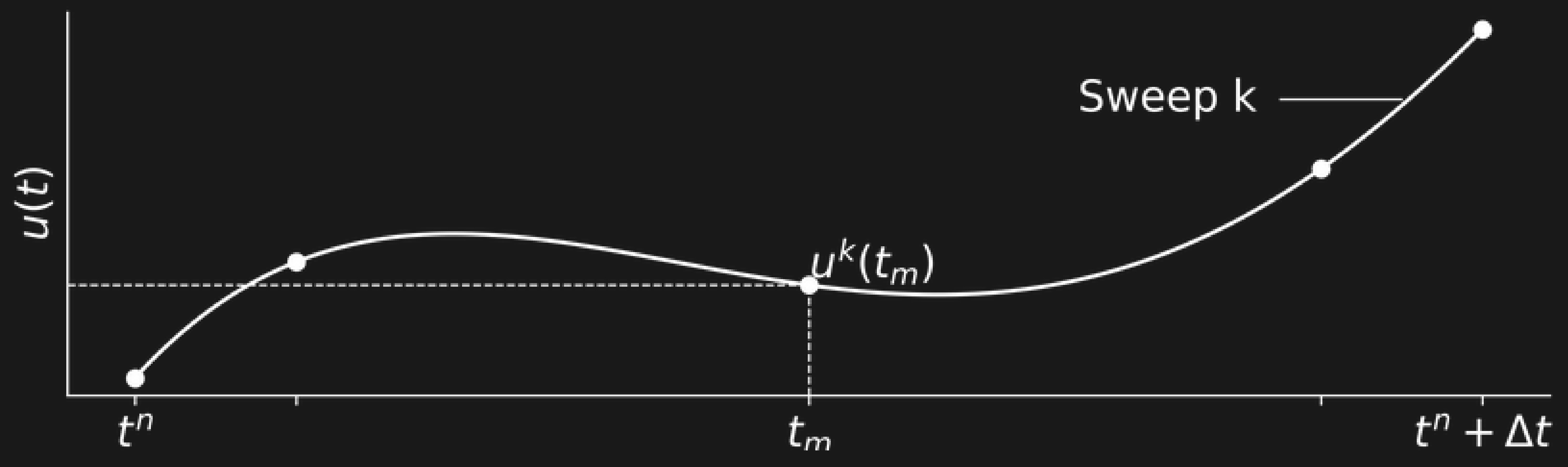
1. SI-MLSDC Methods
2. Dahlquist Problem
3. Conservation Laws
4. Conclusions/ Outlook

SI-MLSDC METHODS

Problem

$$\boxed{\partial_t \mathbf{u} = \mathbf{f}(t, \mathbf{u}(t))}$$

$$\mathbf{u}(t_0) = \mathbf{u}_0$$



collocation points

$$t^n = t_0 < \cdots < t_m < \cdots < t_M = t^n + \Delta t$$

Incremental MLSDC

Predictor

$$\mathbf{u}^0_{l,m} = \mathbf{u}^0_{l,m-1} + \mathbf{H}^0_{l,m} \quad m = 1, \dots M_l$$

Corrector

$$\mathbf{u}^k_{l,m} = \mathbf{u}^k_{l,m-1} + \Delta t \sum_{i=1}^{M_l} w^{NN}_{l,i,m}\, \mathbf{f}(\mathbf{u}^{k-1}_{l,i}, t_{l,i}) + \mathbf{H}^k_{l,m} - \mathbf{H}^{k-1}_{l,m} + g^{NN}_{l,m} \quad m = 1, \dots M_l$$

k = current Sweep, NN = node-to-node and RR collocation points

$$w_{i,m}^{NN} = \int_{\tau_{m-1}}^{\tau_m} \ell_i(\tau)\partial\tau$$

$$\mathbf{H}_m^k \approx \int_{t_{m-1}}^{t_m} \mathbf{f}(\mathbf{u}^k, t)\partial t$$

$$\mathbf{H}_m^k = \begin{cases} \text{IMEX Euler} \\ \text{SI(1)} \\ \text{SI(2)} \end{cases}$$

Matrix Form

$$\underline{\mathbf{F}}_l^{NN}(\underline{\mathbf{u}}_l^k) = \underline{\mathbf{g}}_l^{NN} + \underline{\mathbf{C}}_l^{NN}\left(\underline{\mathbf{H}}_l^k - \underline{\mathbf{H}}_l^{k-1}\right)$$

FAS RHS

Notation

$$\underline{\mathbf{u}}_l^k = \left[\{\mathbf{u}_{l,m}^k\}_{m=1}^{M_l}\right] \text{ for } l = 1, \dots L$$

$$\underline{\mathbf{r}}_l = \underline{\mathcal{R}}_l \underline{\mathbf{r}}_{l+1}$$

$$\underline{\mathbf{v}}_l = \underline{\mathcal{P}}_l \underline{\mathbf{u}}_{l+1}$$

$$\underline{\mathbf{u}}_l = \underline{\mathcal{I}}_l \underline{\mathbf{u}}_{l-1}$$

FAS RHS

$$\underline{\mathbf{g}}_l^{NN} = \begin{cases} \underline{\mathbf{F}}_l^{NN}(\underline{\mathcal{P}}_l \underline{\mathbf{u}}_{l+1}) + \underline{\mathcal{R}}_l \left(\underline{\mathbf{g}}_{l+1}^{NN} - \underline{\mathbf{F}}_{l+1}^{NN}(\underline{\mathbf{u}}_{l+1})\right) & l < L \\ 0 & l = L \end{cases}$$

V-Cycle with L=2

Pre-Smoothing

$\underline{\mathbf{u}}_2 \leftarrow \mathrm{SDC\_Corrector}(\underline{\mathbf{u}}_2, \underline{\mathbf{g}}_2, 1)$

Restriction

$\underline{\mathbf{r}}_2 = \underline{\mathbf{g}}_2 - \underline{\mathbf{F}}_2(\underline{\mathbf{u}}_2)$

$\underline{\mathbf{v}}_1 = \underline{\mathcal{P}}_1 \underline{\mathbf{u}}_2$

$\underline{\mathbf{g}}_1 = \underline{\mathbf{F}}_1(\underline{\mathbf{v}}_1) + \underline{\mathcal{R}}_1 \underline{\mathbf{r}}_2$

Coarse Solution

$\underline{\mathbf{u}}_1 \leftarrow \mathrm{SDC\_Corrector}(\underline{\mathbf{u}}_1, \underline{\mathbf{g}}_1, N)$

Solution Interpolation

$\underline{\mathbf{u}}_2^{\mathrm{new}} = \underline{\mathbf{u}}_2 + \underline{\mathcal{I}}_2(\underline{\mathbf{u}}_1 - \underline{\mathcal{P}}_1 \underline{\mathbf{u}}_2)$

Post-Smoothing

$\underline{\mathbf{u}}_2 \leftarrow \mathrm{SDC\_Corrector}(\underline{\mathbf{u}}_2, \underline{\mathbf{g}}_2, 1)$

Incremental formulation

$$\underline{\mathbf{F}}_l^{NN}(\underline{\mathbf{u}}_l^k) = \underline{\mathbf{g}}_l^{NN} + \underline{\mathbf{C}}_l^{NN}\left(\underline{\mathbf{H}}_l^k - \underline{\mathbf{H}}_l^{k-1}\right)$$

with

$$\underline{\mathbf{C}}_l^{NN} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Non-incremental formulation

$$\underline{\mathbf{F}}_l^{0N}(\underline{\mathbf{u}}_l^k) = \underline{\mathbf{g}}_l^{0N} + \underline{\mathbf{C}}_l^{0N}\left(\underline{\mathbf{H}}_l^k - \underline{\mathbf{H}}_l^{k-1}\right)$$

with

$$\underline{\mathbf{C}}_l^{0N} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

DAHLQUIST

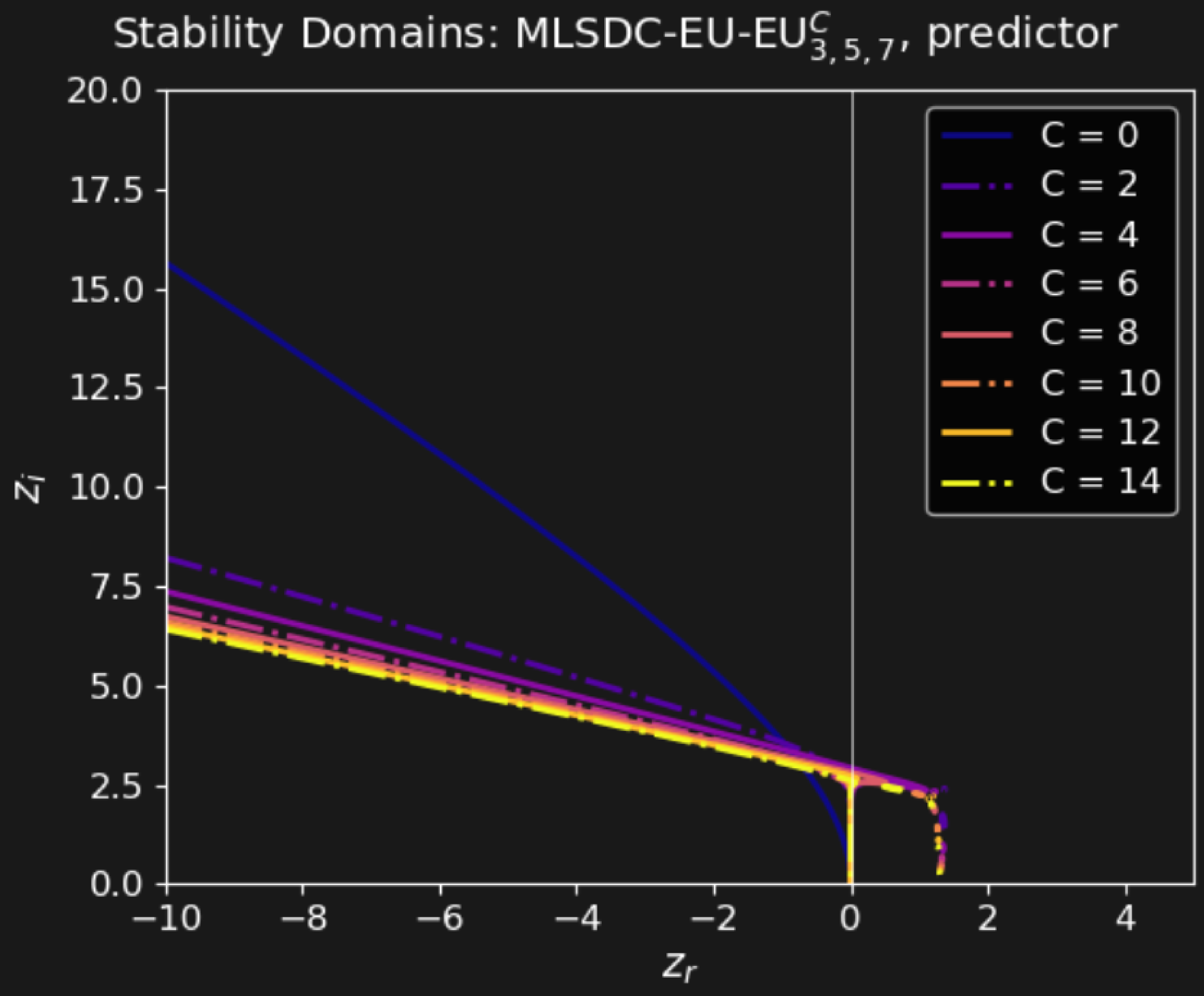
Dahlquist Equation

$$\boxed{\partial_t \hat{u} = \lambda \hat{u}} \text{ with } \lambda \in \mathbb{C}$$

$$\hat{u}(t = 0) = (1, 0), \text{ one time step, } \Delta t = 1$$

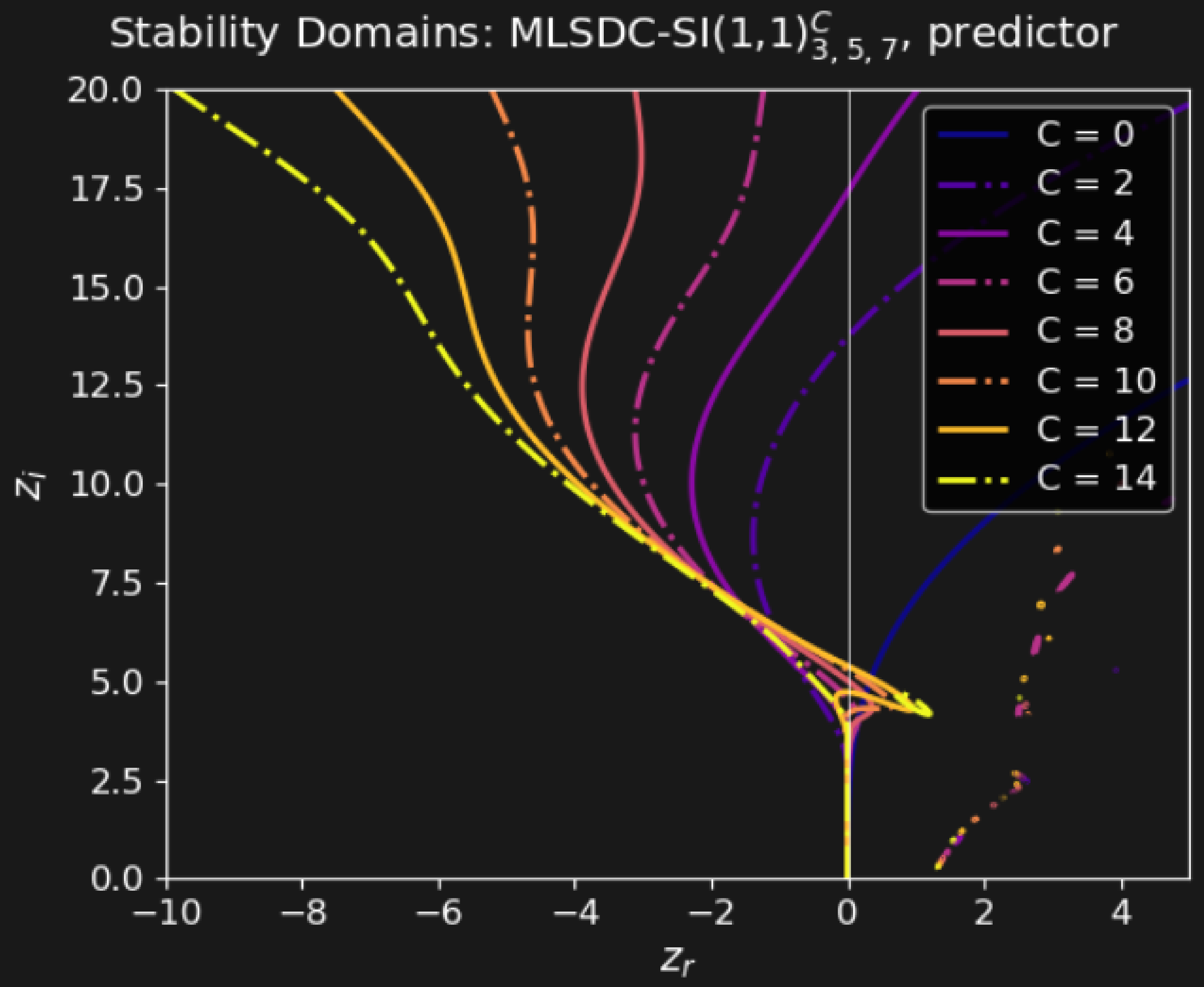
$$\text{one step methods: } R(z) = \frac{\hat{u}^{n+1}}{\hat{u}^n} \rightarrow R(z) = \hat{u}^1 = z$$

Stability MLSDC-EU-EU Incremental

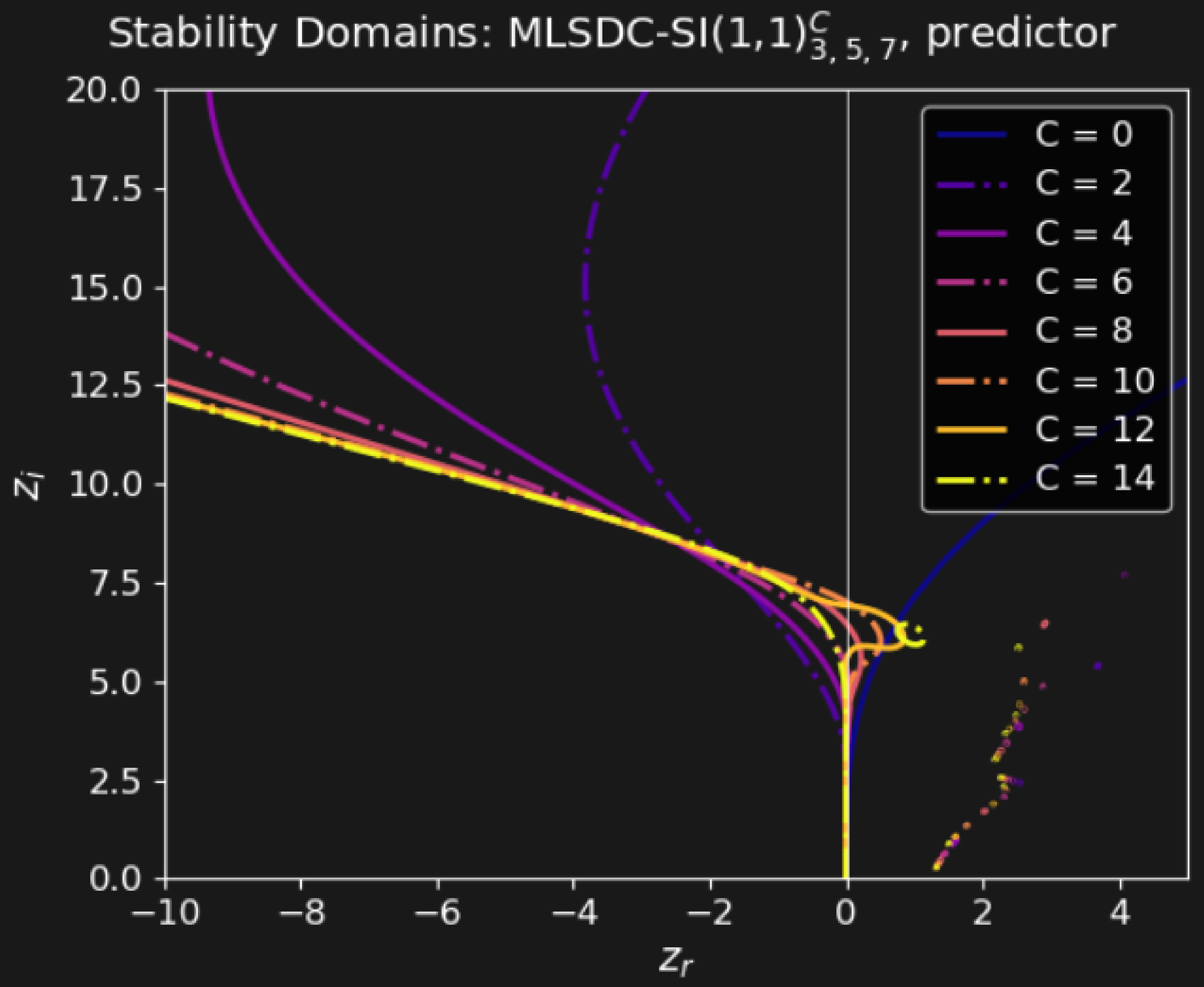


Stability MLSDC-SI(1,1)

Incremental

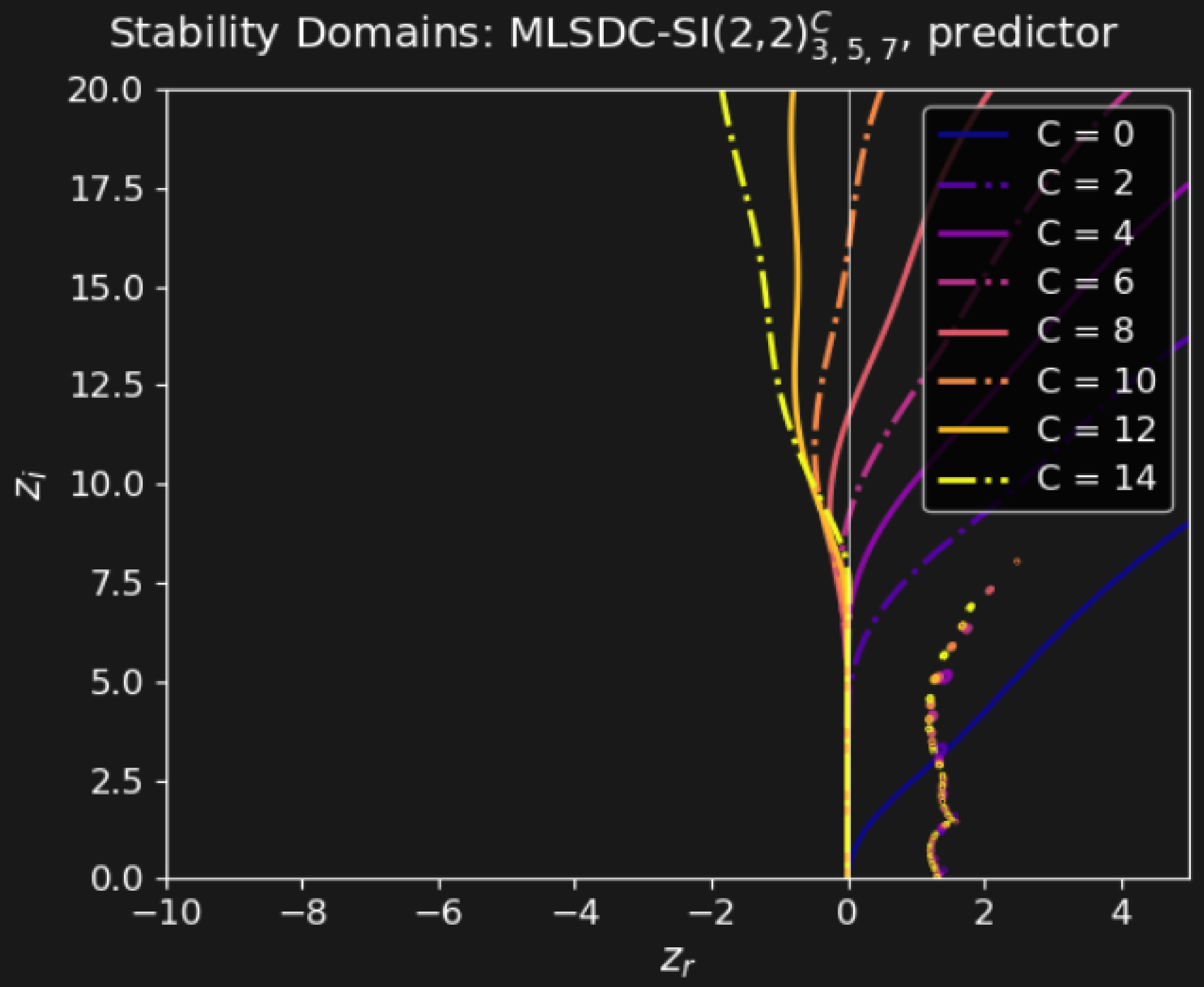


Non-incremental

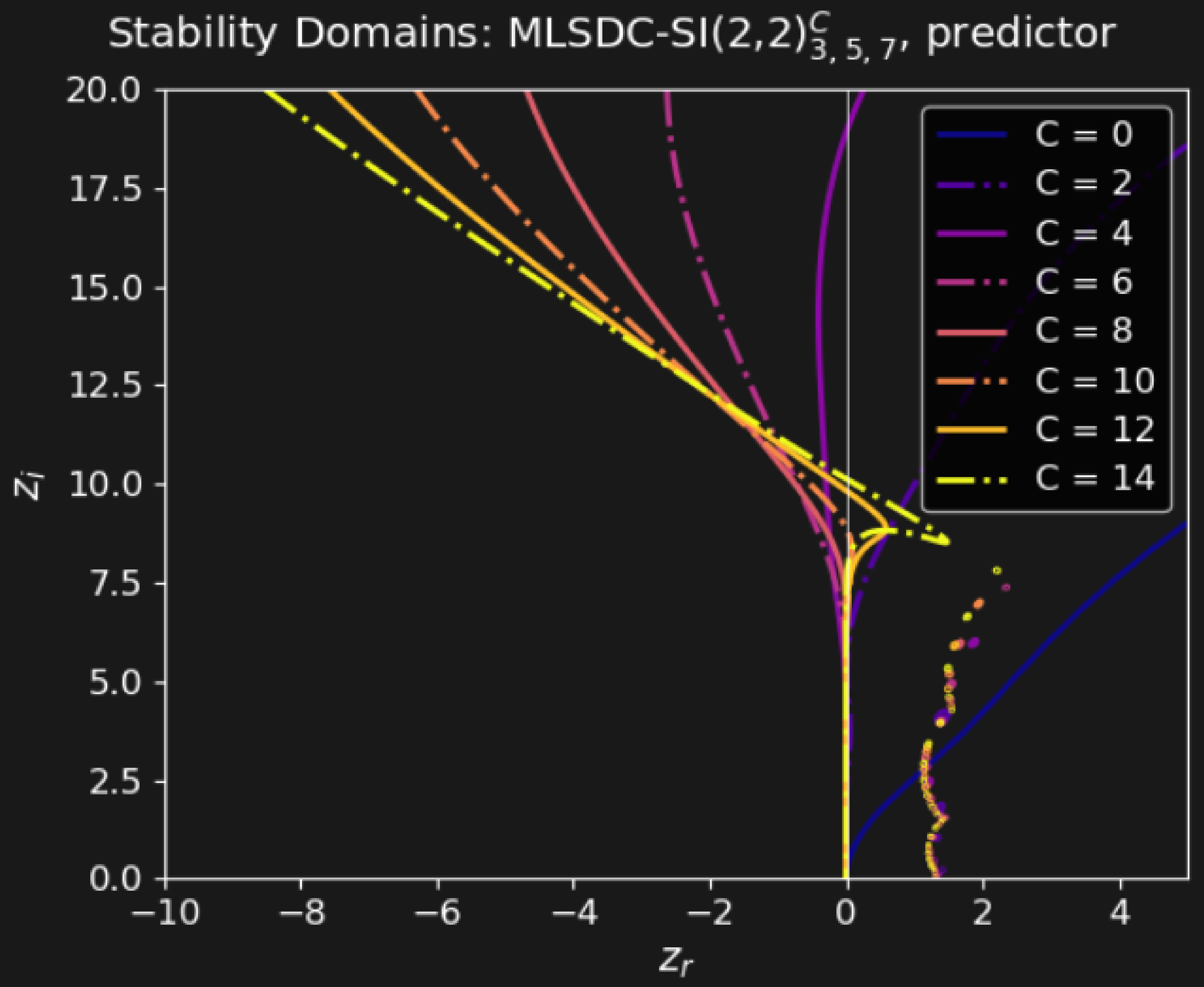


Stability MLSDC-SI(2,2)

Incremental

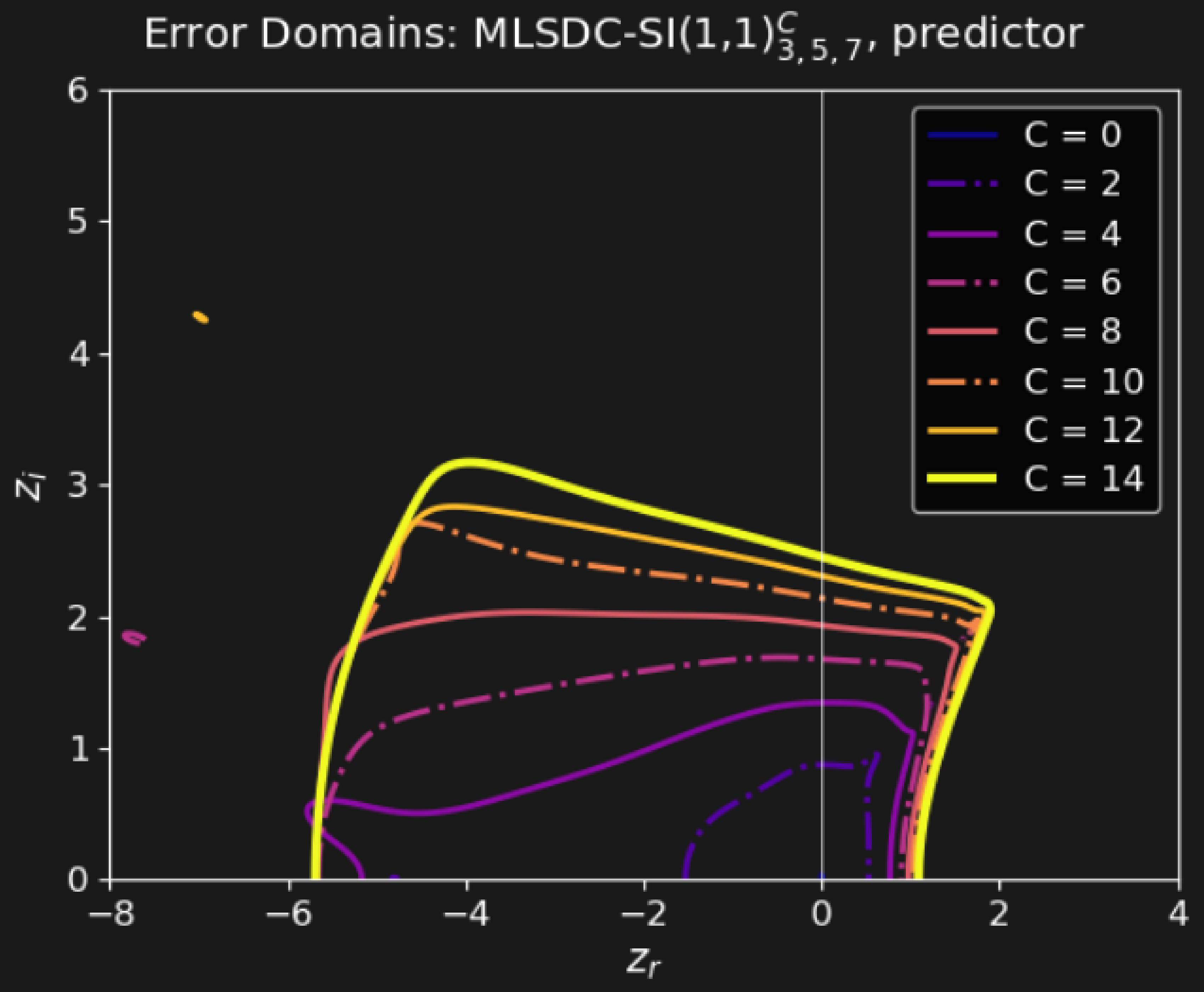


Non-incremental

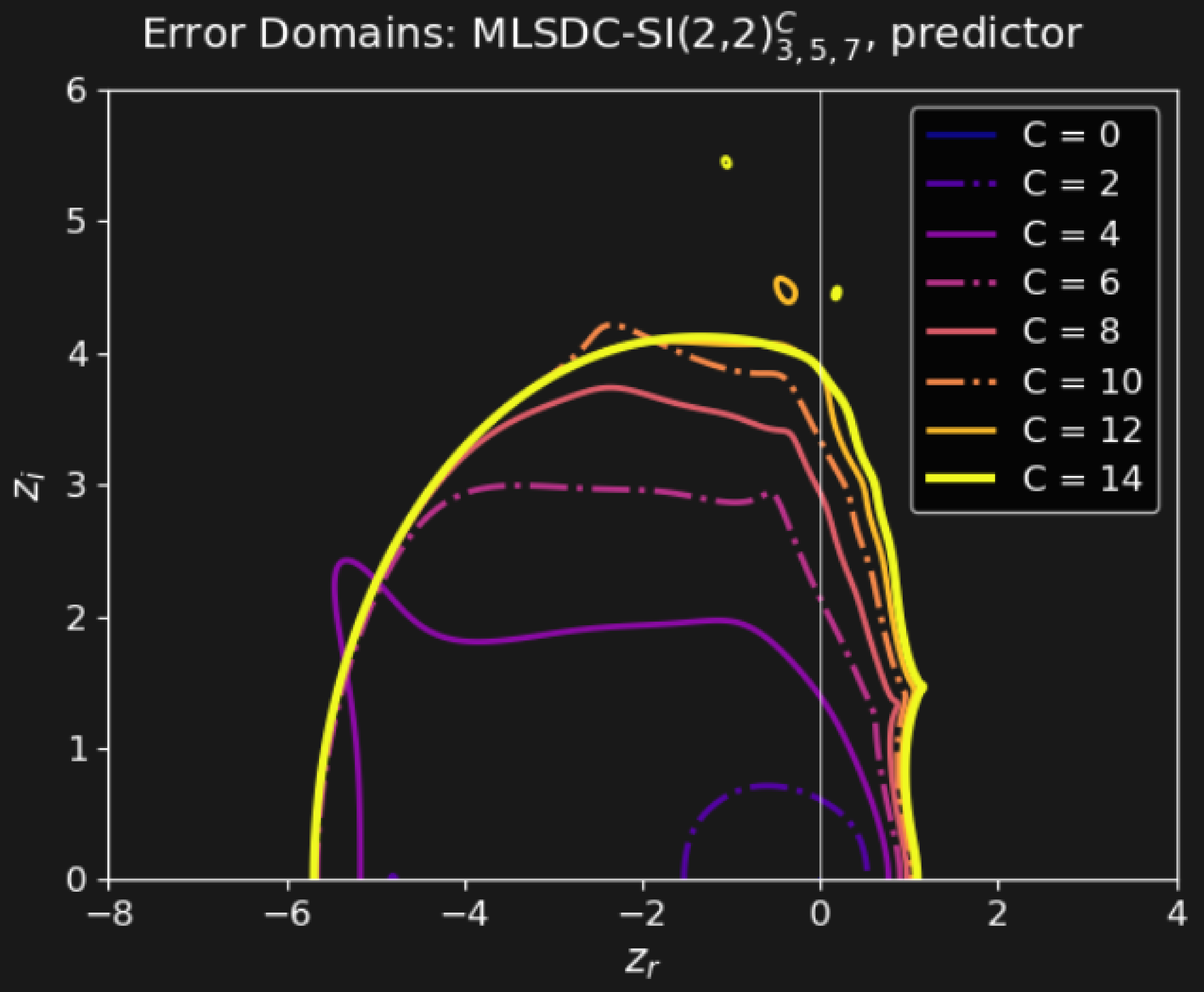


Accuracy MLSDC-SI for $\varepsilon = 10^{-6}$

Incremental MLSDC-SI(1,1)



Incremental MLSDC-SI(2,2)



CONSERVATION LAWS

Spatial Discretization

Discontinuous Galerkin - Spectral Element Method

- Spatial domain with elements $\Omega_h = \cup\Omega^e$
- Elements with polynomial basis of degree P
- Convection term with Riemann flux (scalar) or Roe flux (CNS)
- Diffusion term wit symmetric interior penalty method

Convection-Diffusion Equation

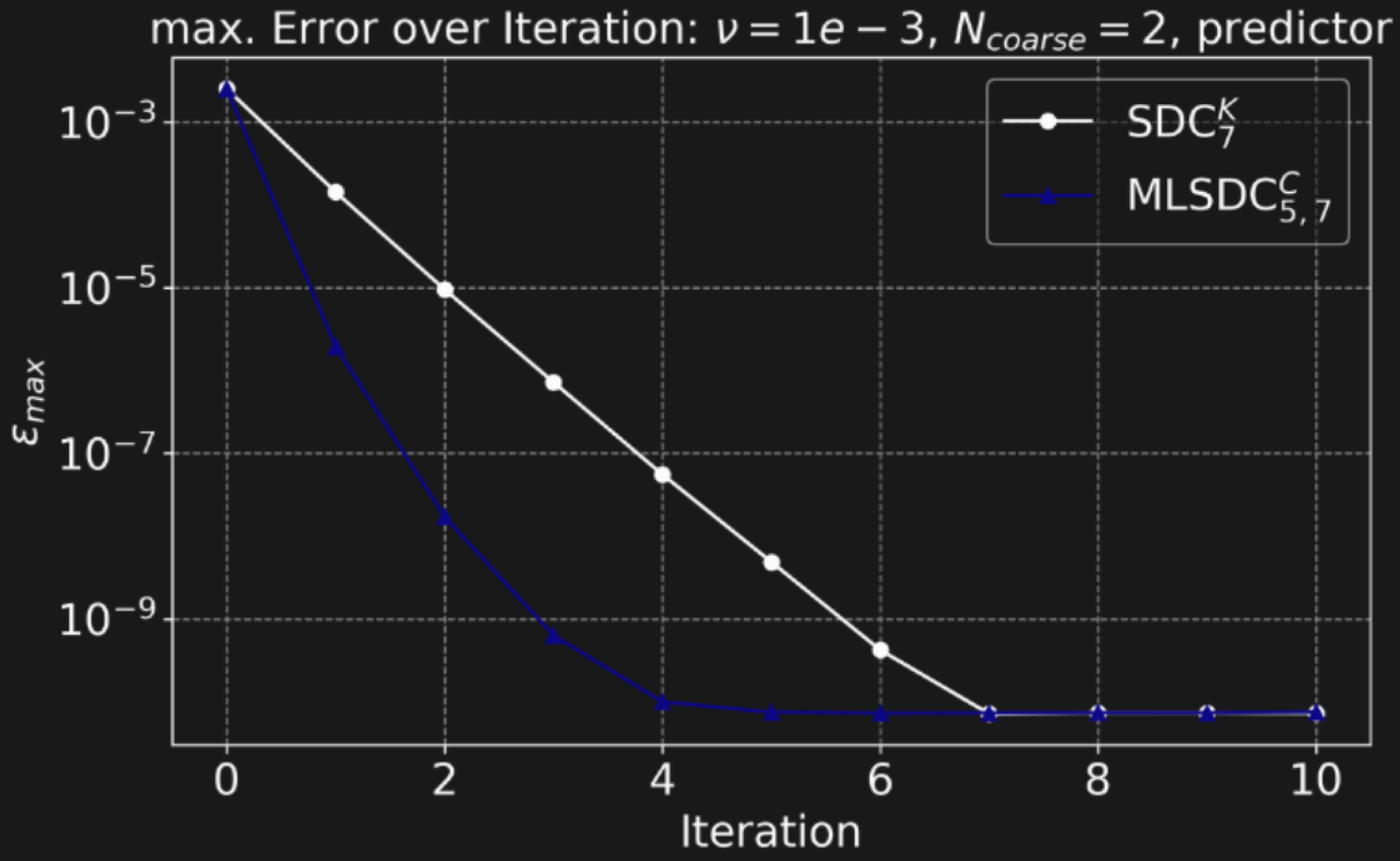
$$\boxed{\partial_t u = -\partial_x(\upsilon u) + \partial_x(\nu \partial_x u)}$$

with constant velocity υ and $\nu > 0$, periodic BC and

$x \in [0, 1], \quad t \in [0, 0.01]$ one time step

	Space		Time	
	$n_{e,l}$	$P_l = Q_l$	$n_{t,l}$	M_l
Level 1	64	10	1	5
Level 2	64	15	1	7

Exact solution - wave package $u(x,t) = \sum_{i=1}^{7} a_i sin\Big(\kappa_i(x - s_i - \upsilon t)\Big) \exp -\kappa_i^2 \nu t$


max. Error over Iteration: $\nu = 1e-3$, $N_{coarse} = 2$, predictor
10^{-3}
10^{-5}
10^{-7}
10^{-9}
ε_{max}
0
2
4
6
8
10
Iteration
SDC_7^K
$MLSDC_{5,7}^C$

Burgers Equation

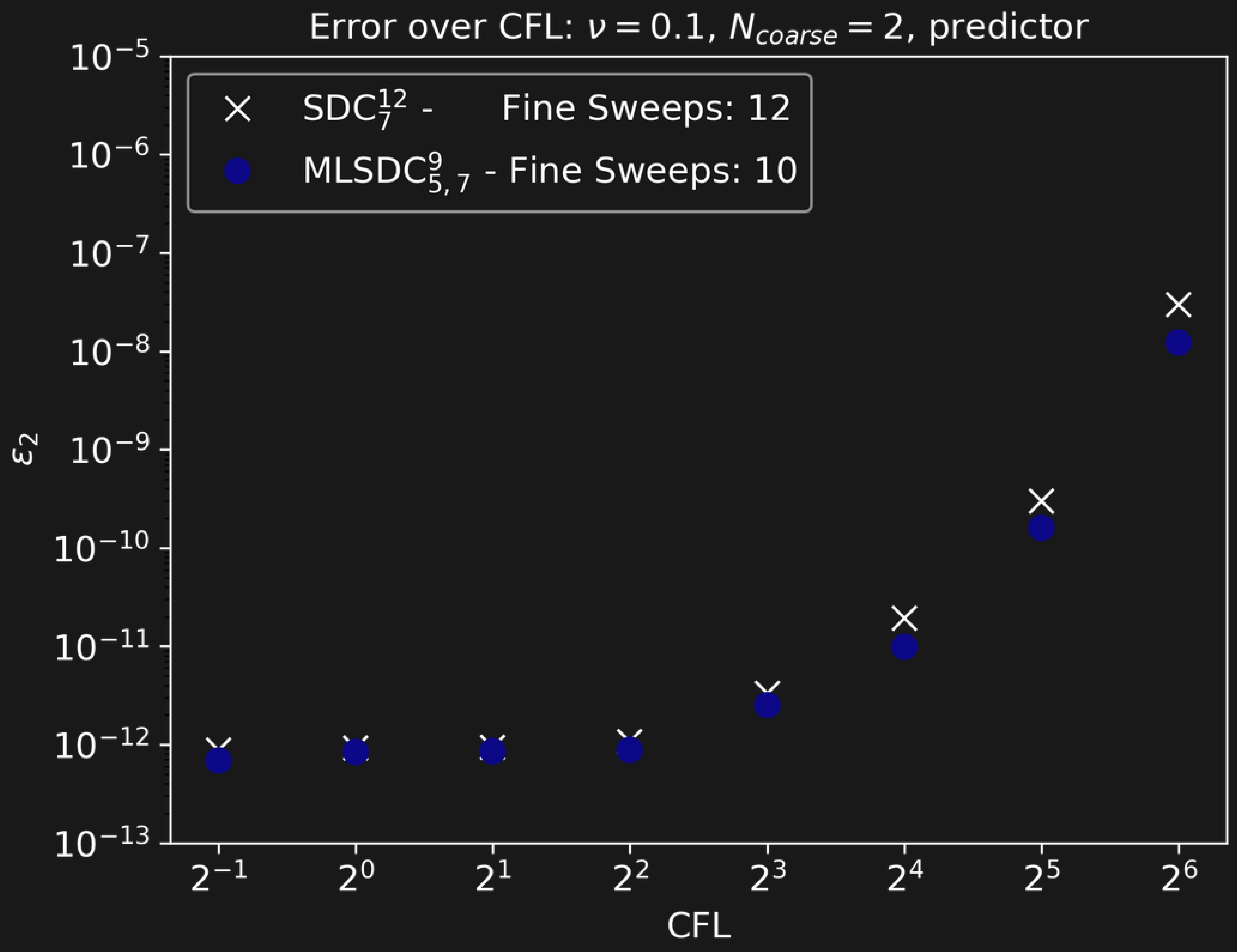
$$\boxed{\partial_t u = -\partial_x\left(\tfrac{1}{2}u^2\right) + \partial_x(\nu\partial_x u) + f_s(x,t)}$$

with $\nu > 0$, Dirichlet and Neumann BC and

$$x \in [-1, 1], \quad t \in [0, 2]$$

	Space			Time	
	$n_{e,l}$	P_l	Q_l	$n_{t,l}$	M_l
Level 1	20	8	12	1	5
Level 2	20	10	15	1	7

Moving front solution: $u(x,t) = 1 - tanh\left(\frac{x+0.5-t}{2\nu}\right)$



Error over CFL: $\nu = 0.1$, $N_{coarse} = 2$, predictor
SDC_7^{12} - Fine Sweeps: 12
$\text{MLSDC}_{5,7}^{9}$ - Fine Sweeps: 10
ε_2
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-13}
2^{-1} 2^{0} 2^{1} 2^{2} 2^{3} 2^{4} 2^{5} 2^{6}
CFL

Compressible Navier Stokes

$$\boxed{\partial_t \mathbf{u} = -\partial_x \mathbf{f}_c(\mathbf{u}) + \partial_x \mathbf{f}_d(\mathbf{u}) + \mathbf{f}_s(\mathbf{u}, x, t)}$$

conservative variables $\mathbf{u}$, convective fluxes $\mathbf{f}_c$ and diffusive fluxes $\mathbf{f}_d$

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho v \\ \rho E \end{bmatrix}, \quad \mathbf{f}_c = \begin{bmatrix} \rho v \\ \rho v^2 + p \\ \rho v E + p v \end{bmatrix}, \quad \mathbf{f}_d = \begin{bmatrix} 0 \\ \frac{4}{3}\eta \partial_x v \\ \frac{4}{3}\eta v \partial_x v + \lambda \partial_x T \end{bmatrix}$$

Acoustic Wave

$$x \in [0, 1], \quad t \in [0, 2.619 * 10^{-2}]$$

Fluid: Air ($R = 287.28, \gamma = 1.4$)

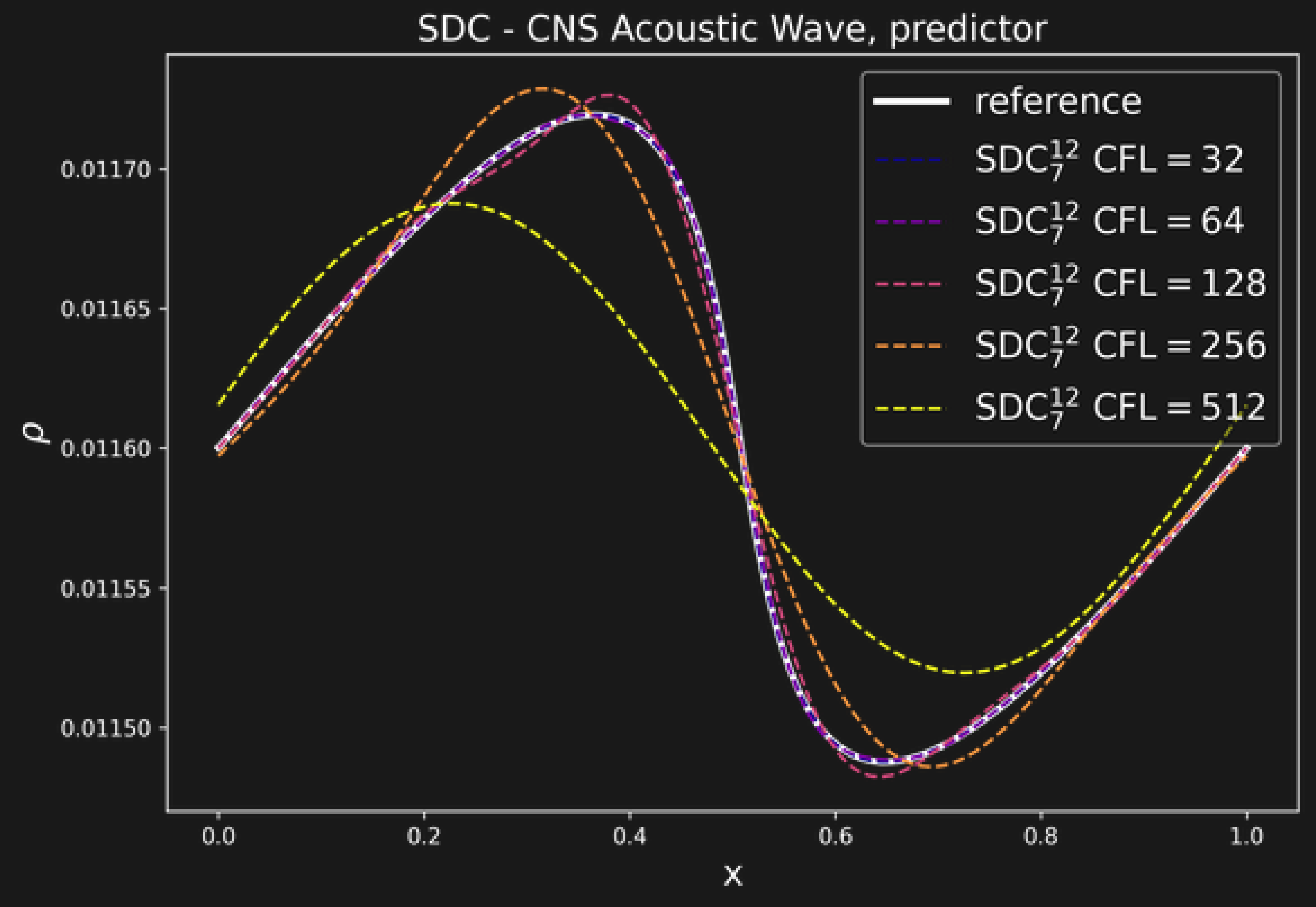
Ma $= 0.1, p_\infty = 1000, T_\infty = 300$, dyn. visc. $\eta = 10^{-5}$, Pr $= 0.75$

	Space			Time	
	$n_{e,l}$	P_l	Q_l	$n_{t,l}$	M_l
Level 1	10	15	30	1	5
Level 2	10	15	30	1	7

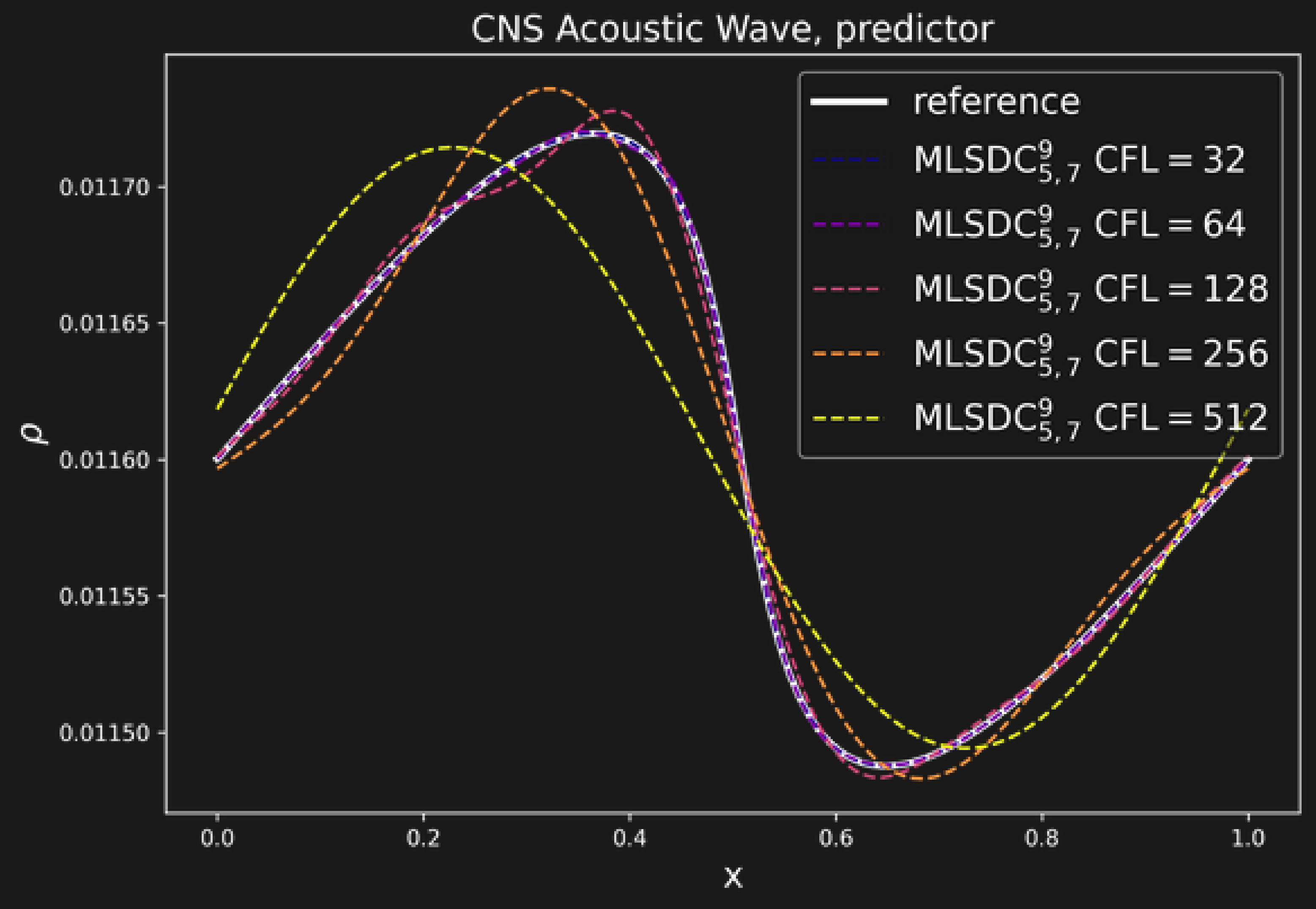
Reference plot with TVD-RK3 and CFL ≈ 0.83

Acoustiv wave SDC vs. MLSDC

SDC



MLSDC



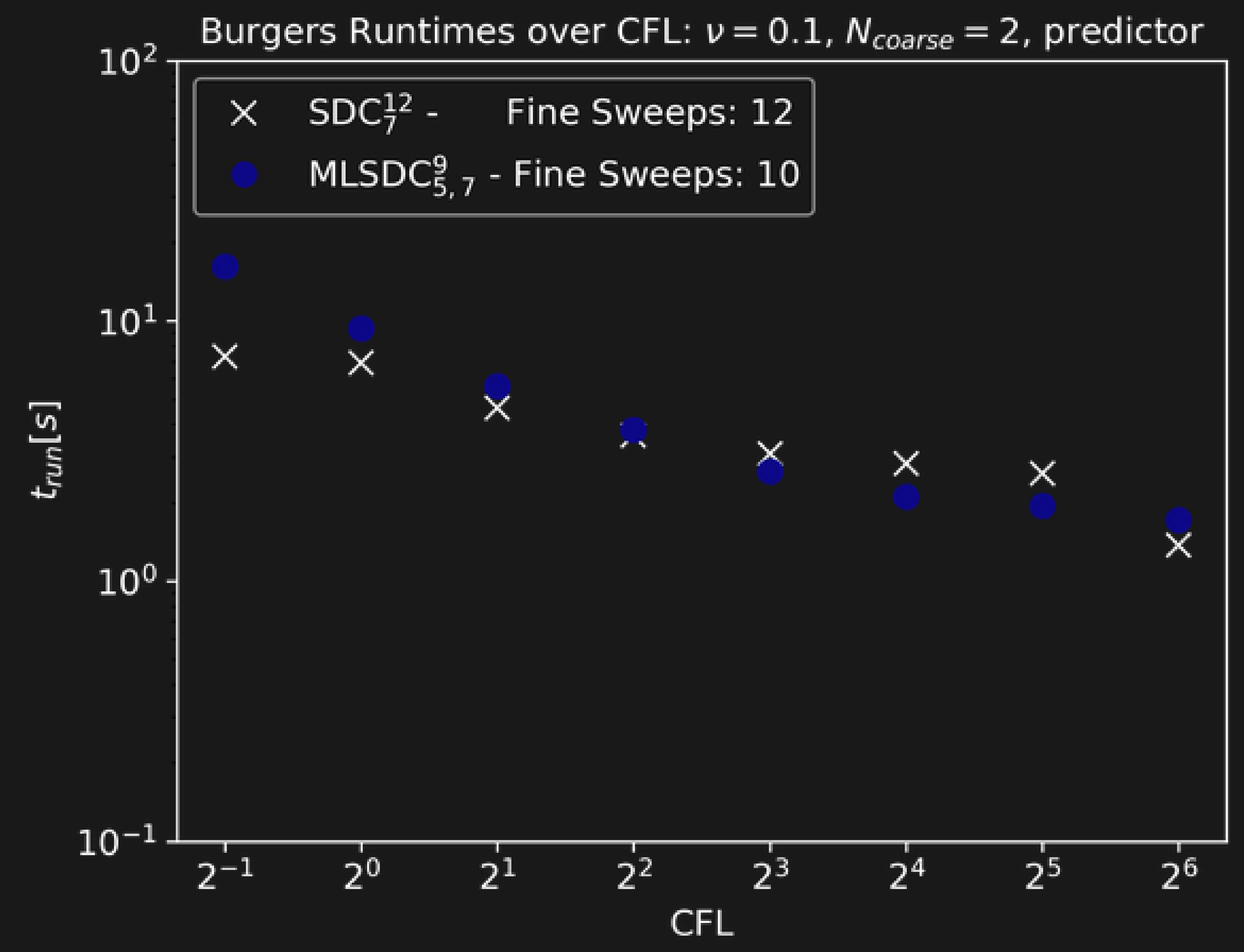
CONCLUSIONS/ OUTLOOK

Conclusions

- Stable SI-MLSDC methods for range of problems
- ML acceleration clearly recognizable
- Prefer to take incremental fromulation

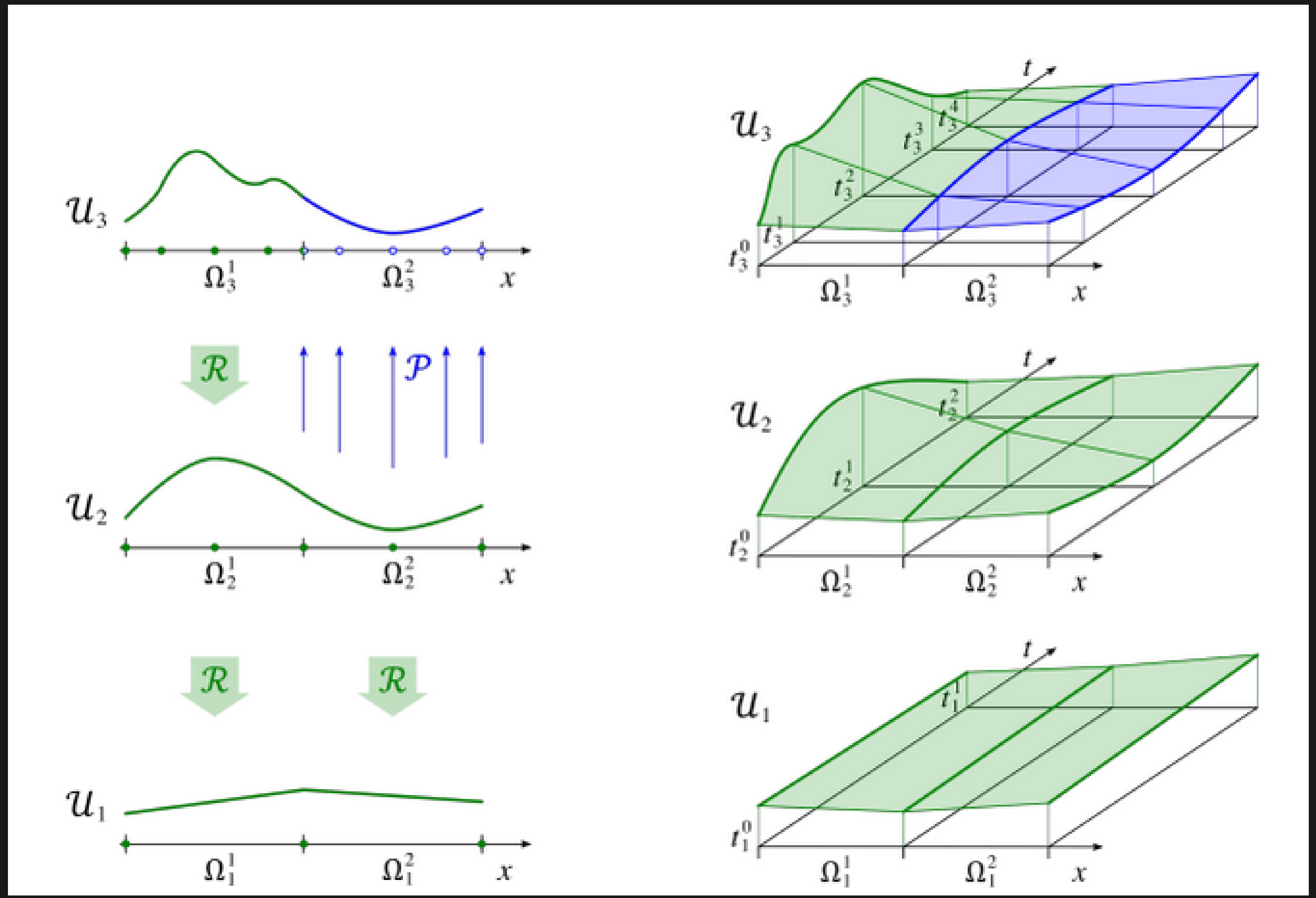
Lessons learned

- Stability slightly worse than for SDC alone
- h-refinement in time problematic
- Projection method plays role in accuracy
- Runtime savings not as high as expected



Outlook

- Runtime analysis Likwid
- 3D $\longrightarrow$ more runtime savings expected
- Multi level adaptive technique



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